Solution exercise set 4

Problem 1

In the previous exercise set we showed that, in the approximation of a point source, the diffusion of the water tracing solution in the pipe follows the solution of the diffusion equation in one dimension

$$\rho(x) = \frac{\text{Cste}}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \tag{1}$$

with $\rho(x)$ the density assumed to be constant over the radial coordinates in the pipe (narrow pipe), D the diffusion coefficient and $\text{Cste} = \frac{M}{A}$. We will use here a more realistic description of the initial concentration profile. For the general case we have

$$\rho(x) = \int_{-\infty}^{\infty} \frac{\rho_0(\xi)}{\sqrt{4\pi Dt}} e^{-\frac{(x-\xi)^2}{4Dt}} d\xi \tag{2}$$

with $\rho_0(\xi)$ the initial concentration profile. All the points of the initial concentration profile contribute (the integral corresponds to the sum of their contributions) to the profile at a time t following the solution of the diffusion equation for a point source (1).

- At $t = 0$, we inject $5ml$ in a pipe with cross section $A = 0.8cm^2$. This gives

$$\rho_0(x) = \begin{cases} \rho_0 = 230\ g/l & |x| < d_0 = 3.125cm \\ 0 & |x| > d_0 = 3.125cm \end{cases}$$

  We have introduced $d_0 = \frac{L}{2}$. Now we can use this result in (2) to get the concentration at any time t

$$\rho(x,t) = \frac{\rho_0}{2}\left[\text{erf}\left(\frac{d_0 - x}{\sqrt{4Dt}}\right) + \text{erf}\left(\frac{d_0 + x}{\sqrt{4Dt}}\right)\right] \tag{3}$$

- The standard deviation σ is given by

$$\sigma^2 = \frac{\int_{-\infty}^{\infty} x^2 \rho(x) dx}{\int_{-\infty}^{\infty} \rho(x) dx} \tag{4}$$

  Now we compute

$$\int_{-\infty}^{\infty} \rho(x) dx = \int_{-\infty}^{\infty} \int_{-d_0}^{d_0} \frac{\rho_0}{\sqrt{4\pi Dt}} e^{-\frac{(x-\xi)^2}{4Dt}} d\xi dx = 2d_0\rho_0. \tag{5}$$

  In the last step we have computed first the integral over x and used the first result of Problem 2 from the previous exercise set.
  We still have to compute

$$\int_{-\infty}^{\infty} x^2 \rho(x) dx = \int_{-\infty}^{\infty} \int_{-d_0}^{d_0} x^2 \frac{\rho_0}{\sqrt{4\pi Dt}} e^{-\frac{(x-\xi)^2}{4Dt}} d\xi dx = \frac{2}{3} d_0^3 \rho_0 + 4 d_0 D t \rho_0 \tag{6}$$

In the last step we have computed first the integral over x making the change of variable $Y = \frac{x-\xi}{\sqrt{4Dt}}$. To compute this integral we used both the first (to compute a term of the type $\int_{-\infty}^{\infty} \exp(-Y^2)$) and the second result of Problem 2 (to compute a term of the type $\int_{-\infty}^{\infty} Y^2 \exp(-Y^2)$) from the previous exercise set. A third term of the form $\int_{-\infty}^{\infty} Y \exp(-Y^2)$ is zero since we integrate an odd function over a domain symmetric upon substitution $Y \leftrightarrow -Y$. Finally we get the result from (4)

$$\sigma = \sqrt{\frac{d_0^2}{3} + 2Dt} \approx 1.8cm \tag{7}$$

for $t = 1s$. The term $2Dt$ gives a negligible contribution to the standard deviation at the time $t = 1s$. However, the term d_0^2 is constant over time. This means after a sufficiently long time, the contribution of the term $2Dt$ will become dominant (for our problem this time is typically really long, see the last result of Problem 1 of the previous exercise set). The point source approximation is a good approximation as long as we are interested at points in the pipe at a distance much larger than the domain L occupied by the injected fluid at $t = 0$.

- The solution in the presence of advection can be obtained replacing x by $x - v_{\rm adv}t$ in (2). The integral can be computed

$$\rho(x) = \int_{-\infty}^{\infty} \frac{\rho_0(\xi)}{\sqrt{4\pi Dt}} e^{-\frac{(x-(\xi+v_{\rm adv}t))^2}{4Dt}} d\xi = \frac{\rho_0}{2}\left[\mathrm{erf}(\frac{d_0 - x + v_{\rm adv}t}{\sqrt{4Dt}}) + \mathrm{erf}(\frac{d_0 + x - v_{\rm adv}t}{\sqrt{4Dt}})\right] \tag{8}$$

  From this solution we also get immediately the solution in the absence of advection (3) setting $v_{\rm adv} = 0$.

- Advection can be neglected as long as the length scale related to advection is smaller that all the other relevant length scales of the system (carefull, only quantities with the same dimension can be compared !!). For our problem we need to compare three length to decide which part of the process is dominant or can be neglected. At a time t we have the three lengths to compare

$$d_0 \quad , \quad v_{\rm adv}t \quad \text{and} \quad \sqrt{Dt}. \tag{9}$$

  The first length is fixed while the others evolve in time. If we compare only advection and diffusion, we recover the condition (see the lecture) that the diffusion is dominant if the Péclet number $P_{\rm e} = \frac{D}{v_{\rm adv}^2 t}$ is large.